DEFORMATION OF DIRAC STRUCTURES ALONG ISOTROPIC SUBBUNDLES

IVÁN CALVO, FERNANDO FALCETO, AND MARCO ZAMBON

Abstract. Given a Dirac subbundle and an isotropic subbundle of a Courant algebroid, we provide a canonical method to obtain a new Dirac subbundle. When the original Dirac subbundle is involutive (i.e., a Dirac structure) this construction has interesting applications, for instance to Dirac's theory of constraints and to the Marsden-Ratiu reduction in Poisson geometry.

1. Introduction

The concept of *Dirac structure* generalizes Poisson and presymplectic structures by embedding them in the framework of the geometry of $TM \oplus T^*M$ or, more generally, the geometry of a *Courant algebroid*. Dirac structures were introduced in a remarkable paper by T. Courant [7]. Therein, they are related to the Marsden-Weinstein reduction [14] and to the Dirac bracket [9] on a submanifold of a Poisson manifold. More recently, Dirac structures have been considered in connection to the reduction of implicit Hamiltonian systems (see [2],[1]). This simple but powerful structure allows to deal with mechanical situations in which we have both gauge symmetries and Casimir functions.

We present a construction which takes an isotropic subbundle S and a Dirac subbundle D of an exact Courant algebroid, and produces a new Dirac subbundle D^S (Def. 3.1). This construction, which we refer to as *stretching*, was introduced by the first two authors in [6]. When both S and D are involutive, we find conditions ensuring that D^S is also involutive, i.e. a Dirac structure (Thm. 4.1).

We further show that three prominent classes of Dirac structures are indeed stretched Dirac structures: the Dirac brackets that appeared in Dirac's theory of constraints, the Dirac structures underlying the Marsden-Ratiu quotients in Poisson geometry [13], and coupling Dirac structures on Poisson fibrations [3].

The paper is organized as follows. In Section 2 we review basic definitions, in Section 3 we describe our stretching construction, in Section 4 we discuss when the stretched structure is involutive, and in Section 5 we present examples and applications.

Date: October 14, 2014.
Keywords: Courant Algebroid. Dirac structure. Dirac's theory of constraints. Reduction.
2000 MSC: 53D17, 53D99, 70H45.

2. Courant algebroids and Dirac structures

Definition 2.1. A **Courant algebroid** [12] over a manifold M is a vector bundle $E \to M$ equipped with an $\mathbb{R}$-bilinear bracket $[\cdot,\cdot]$ on $\Gamma(E)$, a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on the fibers and a bundle map $\pi : E \to TM$ (the *anchor*) satisfying, for any $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

(i) $[e_1,[e_2,e_3]] = [[e_1,e_2],e_3] + [e_2,[e_1,e_3]]$
(ii) $\pi([e_1,e_2]) = [\pi(e_1),\pi(e_2)]$
(iii) $[e_1, fe_2] = f[e_1,e_2] + (\pi(e_1)f)e_2$
(iv) $\pi(e_1)\langle e_2,e_3\rangle = \langle [e_1,e_2],e_3\rangle + \langle e_2,[e_1,e_3]\rangle$
(v) $[e,e] = \mathcal{D}\langle e,e\rangle$

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined by $\mathcal{D} = \frac{1}{2}\pi^* \circ \mathrm{d}$, using the bilinear form to identify E and its dual.

We see from axiom (v) that the bracket is not skew-symmetric, but rather satisfies $[e_1,e_2] = -[e_2,e_1] + 2\mathcal{D}\langle e_1,e_2\rangle$.

A Courant algebroid is called **exact** if

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0$$

is an exact sequence. Choosing a splitting $TM \to E$ of the above sequence with isotropic image allows one to identify the exact Courant algebroid with $TM \oplus T^*M$ endowed with the natural symmetric pairing

$$\langle (X,\xi),(X',\xi')\rangle = \frac{1}{2}(i_{X'}\xi + i_X\xi')$$

and the Courant bracket

$$[(X,\xi),(X',\xi')] = ([X,X'], \mathcal{L}_X\xi' - i_{X'}\mathrm{d}\xi + i_{X'}i_X H)$$

for some closed 3-form H. In fact, the Courant algebroid uniquely determines the cohomology class of H, called Ševera class. The anchor π is given by the projection onto the first component. When it is important to stress the value of the 3-form H we shall use the notation E_H for $TM \oplus T^*M$ equipped with this Courant algebroid structure.

Definition 2.2. A **Dirac subbundle** or **almost Dirac structure** in an exact Courant algebroid E is a subbundle $D \subset E$ which is maximal isotropic with respect to $\langle\cdot,\cdot\rangle$. The maximal isotropicity condition implies that $D^\perp = D$, where $D^\perp$ stands for the orthogonal subspace of D. In particular, $\mathrm{rank}(D) = \dim(M)$.

A **Dirac structure** is an involutive Dirac subbundle, i.e. a Dirac subbundle D whose sections closed under the Courant bracket. In this case the restriction to D of the Courant bracket is skew-symmetric and D with anchor π is a Lie algebroid.

The two basic examples of Dirac structures are:

Example 2.1. For any 2-form ω, the graph L_ω of $\omega^\flat : TM \to T^*M$ is a Dirac subbundle such that $\pi(L_\omega) = TM$. L_ω is a Dirac structure in E_H if and only if $\mathrm{d}\omega = -H$. In particular, L_ω is a Dirac structure in E_0 if and only if ω is closed.

Example 2.2. Let Π be a bivector field on M. The graph L_Π of the map $\Pi^\sharp : T^*M \to TM$ is always a Dirac subbundle. In this case the natural projection from L_Π to T^*M is one-to-one. L_Π is a Dirac structure in E_H if and only if Π is a twisted Poisson structure. In particular, L_Π is a Dirac structure in E_0 if and only Π is a Poisson structure.

3. Stretched Dirac subbundles

Until the end of this note we will assume the following setup:

$$\boxed{\begin{array}{c} E \text{ is an exact Courant algebroid} \\ S \subset E \text{ is an isotropic subbundle (i.e. } S \subset S^\perp) \\ D \subset E \text{ is an Dirac subbundle.} \end{array}}$$

We further assume that $D \cap S$ (or equivalently $D \cap S^\perp$) has constant rank along M.

Definition 3.1. The **stretching of D along** S [6] is the Dirac subbundle

$$D^S := (D \cap S^\perp) + S.$$

To justify the fact that D^S is maximal isotropic we use

$$\begin{aligned} (D^S)^\perp &= (D^\perp + S) \cap S^\perp \\ &= (D \cap S^\perp) + S = D^S, \end{aligned}$$

where in the last line we have used that D is maximal isotropic and S is a subset of $S^\perp$. It is also clear that D^S, as the sum of two subbundles whose intersection has constant rank, is a (smooth) subbundle.

D^S is the Dirac subbundle closest to D among those containing S, as stated in the following

Proposition 3.1. [6] *Let D, S and D^S be as above and let D' be a Dirac subbundle such that $S \subset D'$. Then, $D' \cap D \subset D^S \cap D$. In addition, $D' \cap D = D^S \cap D$ if and only if $D' = D^S$.*

Proof. From the isotropicity of D' and $S \subset D'$ we deduce that $D' \subset S^\perp$. Hence,

$$D' \cap D \subset S^\perp \cap D = D^S \cap D.$$

If the equality $D' \cap D = D^S \cap D$ holds, then $D' \supset D' \cap D = S^\perp \cap D$. Since $S \subset D'$, we find that $D^S = (D \cap S^\perp) + S \subset D'$. But D^S and D' have the same dimension, so that they are equal. $\square$

4. Integrability

In this section we determine various properties of D^S, in particular conditions under which D^S is a Dirac structure.

Lemma 4.1. *Assume that S and D are closed under the Courant bracket. Then the set of S-invariant sections of D^S*

$$\{e \in D^S : [\Gamma(S), e] \subset \Gamma(S)\}$$

is closed under the Courant bracket.

Proof. Consider two sections $e_1, e_2 \in \Gamma(D^S)$ which are S-invariant, i.e., $[\Gamma(S), e_i] \in \Gamma(S)$. First, let us prove that $[e_1, e_2]$ is an S-invariant section. Take $s \in \Gamma(S)$ and write

$$[s, [e_1, e_2]] \quad = \quad [[s, e_1], e_2] + [e_1, [s, e_2]]$$

by Def. 2.1 i). Now recall that $[e, s] = -[s, e]$ for $e \in \Gamma(D^S)$ and $s \in \Gamma(S)$ because $D^S = (D + S) \cap S^\perp \subset S^\perp$. The S-invariance of $[e_1, e_2]$ follows immediately.

Next we show that $[e_1, e_2] \in \Gamma(D^S)$. Since we assumed that both $D \cap S^\perp$ and S are subbundles, every section $e \in \Gamma(D^S)$ can be written as $e = v + w$ with $v \in \Gamma(D \cap S^\perp)$ and $w \in \Gamma(S)$. Notice that if e is S-invariant, v is also S-invariant because S is Courant involutive. The expression

$$[e_1, e_2] = [v_1 + w_1, v_2 + w_2] = [v_1, v_2] + [v_1, w_2] + [w_1, v_2] + [w_1, w_2] \tag{4.1}$$

makes clear that $[e_1, e_2] \in \Gamma(D + S)$, since $[v_1, v_2] \in \Gamma(D)$ and the remaining terms on the right-hand side of eq. (4.1) are sections of S. To prove that $[e_1, e_2] \in \Gamma(S^\perp)$ notice that, for any $s \in \Gamma(S)$,

$$\langle s, [e_1, e_2] \rangle \quad = \quad \pi(e_1)\langle s, e_2 \rangle - \langle [e_1, s], e_2 \rangle = 0$$

where we have used Def. 2.1 iv) as well as the orthogonality of s and e_i, $i = 1, 2$. □

Inspired by [15] we will give the following

Definition 4.1. Given a Dirac subbundle D and an involutive isotropic subbundle $S \subset E$, we say that S is **canonical** for D if there exists a local S-invariant section[1] of D^S passing through any of its points.

Proposition 4.1. *Assume that S and D are closed under the Courant bracket. We have the following chain of implications:*

a) *S is canonical for D $\Rightarrow$*
b) *D^S is a Dirac structure $\Rightarrow$*
c) *$[\Gamma(S), \Gamma(D^S)] \subset \Gamma(D^S)$ (i.e. S preserves D^S)*

Proof. a) $\Rightarrow$ b): We have to show that the Courant bracket of two sections $v, v' \in \Gamma(D^S)$ is again a section of D^S. We write v and v' in terms of a local basis, $\{e_i\}$, of S-invariant sections (such a basis always exists due to the canonicity of S). From Def. 2.1 iii) and Lemma 4.1 one immediately obtains that $[v, v']$ is a linear combination of the e_i's and hence belongs to $\Gamma(D^S)$.

b) $\Rightarrow$ c): holds because $S \subset D^S$. □

The following theorem gives sufficient conditions to ensure that D^S is a Dirac structure.

Theorem 4.1. *Assume that S and D are closed under the Courant bracket and additionally that $\pi(S^\perp)$ is an integrable regular distribution. Then items a),b),c) of Prop. 4.1 are all equivalent.*

[1]Recall that a section e is S-invariant iff $[\Gamma(S), e] \subset \Gamma(S)$.

Proof. We just need to show c) $\Rightarrow$ a) in Prop. 4.1. Notice first that $\pi(S)$ is a regular integrable distribution. Indeed

$$\mathrm{Ker}(\pi) \cap S = \pi^*(\pi(S^\perp)^\circ),$$

and given that $\pi(S^\perp)$ is regular and π^* is injective for exact Courant algebroids, it follows that $\mathrm{Ker}(\pi) \cap S$ is a subbundle. Now, the fact that S is also a subbundle implies the regularity of $\pi(S)$. Integrability follows from the assumption that $\Gamma(S)$ is closed under the Courant bracket.

Take a commuting basis of local sections of $\pi(S)$ denoted by $\{\partial_i\}$. Fix lifts s_i of ∂_i to S, i.e. $s_i \in \Gamma(S)$ and $\pi(s_i) = \partial_i$. Since we are assuming c) of Prop. 4.1 we can define a partial $\pi(S)$-connection on D^S by imposing

$$\nabla_i e := [s_i, e].$$

The involutivity of S and Def. 2.1 ii) imply that $[\Gamma(S), \Gamma(\mathrm{Ker}(\pi) \cap S)] \subset \Gamma(\mathrm{Ker}(\pi)\cap S)$, so we can use ∇ to define a partial $\pi(S)$-connection[2] $\widetilde{\nabla}$ on $D^S/(\mathrm{Ker}(\pi)\cap S)$. We now argue that $\widetilde{\nabla}$ is flat.

The curvature of the connection ∇, with components F_{ij}, is given by

$$F_{ij} e = \nabla_i \nabla_j e - \nabla_j \nabla_i e = [s_i[s_j, e]] - [s_j[s_i, e]] = [[s_i, s_j], e], \tag{4.2}$$

and given that ∂_i and ∂_j commute and S is involutive we have $[s_i, s_j] \in \mathrm{Ker}(\pi) \cap S$.

Next we want to show that

$$[\Gamma(\mathrm{Ker}(\pi) \cap S), \Gamma(D^S)] \subset \Gamma(\mathrm{Ker}(\pi) \cap S). \tag{4.3}$$

For that, take a section $s \in \Gamma(\mathrm{Ker}(\pi)\cap S)$ and write $s = \pi^*(\eta)$ with $\eta \in \Gamma(\pi(S^\perp)^\circ)$. Also take arbitrary sections $e \in \Gamma(D^S)$ and $s^\perp \in \Gamma(S^\perp)$. Now

$$\begin{aligned}
\langle [s, e], s^\perp \rangle &= \langle \pi^*(\eta), [e, s^\perp] \rangle - \pi(e)\langle s, s^\perp \rangle \\
&= i_{\pi([e, s^\perp])} \eta \\
&= i_{[\pi(e), \pi(s^\perp)]} \eta \\
&= 0,
\end{aligned}$$

where in the first equality we used Def. 2.1 iv) and in last equality we used that $D^S \subset S^\perp$ and $\pi(S^\perp)$ is integrable.

Eq. (4.2) and eq. (4.3) together imply that $\widetilde{\nabla}$ is a flat connection. Hence through any point of $D^S/(\mathrm{Ker}(\pi) \cap S)$ passes a local horizontal sections for $\widetilde{\nabla}$, and any lift of it to a section e_h of D^S satisfies $\nabla_i e_h \in \Gamma(\mathrm{Ker}(\pi) \cap S)$ for all i. But the sections $\{s_i\}$ used to build the connection ∇, together with $\Gamma(\mathrm{Ker}(\pi)\cap S)$, span $\Gamma(S)$. Hence using Def. 2.1 iii) and eq. (4.3) we get

$$[\Gamma(S), e_h] \subset \Gamma(S),$$

completing the proof. $\square$

Remark 4.1. With the hypotheses of Theorem 4.1, if $[\Gamma(S), \Gamma(D)] \subset \Gamma(D)$ then S is canonical for D (because condition c) in Prop. 4.1 is satisfied). The converse is not true, see e.g. [15] for a counterexample in the context of Poisson manifolds.

[2]The connection $\widetilde{\nabla}$ depends on the choice of lifts $s_i \in \Gamma(S)$.

5. Examples and applications

In this section we work only with the exact Courant algebroid E_0. The first two examples show that two well-known constructions in Poisson geometry, once phrased in tensorial terms, correspond to the stretching of Poisson structures.

5.1. **Dirac brackets.** We give a description of the classical Dirac bracket in tensorial terms, i.e. in terms of Dirac structures, thereby giving a clear geometric interpretation to the Dirac bracket. Further we present a natural generalization.

We recall first the construction of the Dirac bracket on a Poisson manifold (M, Π).

Definition 5.1. Given a regular foliation R on an open set $U \subset M$ whose leaves N are the level sets of second class constraints $\varphi^1, \dots, \varphi^m$ (i.e. independent functions for which the matrix $C^{ab} := \{\varphi^a, \varphi^b\}_\Pi$ is invertible, with inverse C_{ab}), the **Dirac bracket** is defined as

$$\{f, g\}_{Dirac} := \{f, g\}_\Pi - \{f, \varphi^a\}_\Pi C_{ab} \{\varphi^b, g\}_\Pi. \tag{5.1}$$

We denote by Π_{Dirac} the bivector field corresponding to the bracket $\{\cdot, \cdot\}_{Dirac}$.

Lemma 5.1. *i) The level sets N of φ are* cosymplectic *submanifolds of* (M, Π) *and therefore have a Poisson structure induced by* Π.

ii) (M, Π_{Dirac}) *is obtained putting together the level sets N of φ, endowed with the Poisson structure induced by* Π. *In particular the Dirac bracket* (5.1) *depends only on the level sets of the constraints (and not on the constraints themselves).*

Remark 5.1. 1) Here and in the following we use repeatedly the following fact: a Poisson (Dirac) manifold is determined by its foliation into symplectic (presymplectic) leaves.
2) Lemma 5.1 ii) recovers the fact that (5.1) is a Poisson bracket.

Proof. i) Since the φ^i are second class constraints, the leaves N of R satisfy $\Pi^\sharp TN^\circ \oplus TN = TM|_N$, which by definition means that they are cosymplectic submanifolds. There is an induced Poisson structure on N [8, Sect. 8], obtained pulling back to N the Dirac structure given by the graph of Π. The corresponding Poisson bracket of functions f, g on N is $\{\tilde{f}, \tilde{g}\}_\Pi|_N$, where $\tilde{f}, \tilde{g}$ are extensions of f, g to M and $d\tilde{f}$ is required to annihilate $\Pi^\sharp(TN^\circ)$ at points of N.

ii) One checks easily that $\{\varphi^i, g\}_{Dirac} = 0$ for all $g \in C^\infty(U)$, i.e. that the φ^i are Casimir functions for Π_{Dirac}, hence the level sets N of φ are Poisson submanifolds (i.e. unions of symplectic leaves) w.r.t. Π_{Dirac}. The Poisson structure on N as a Poisson submanifold of (M,Π_{Dirac}) agrees with the one induced by Π in the way described in i). Indeed for all functions f, g on N and extensions $\tilde{f}, \tilde{g}$ as above we have $\{\tilde{f}, \tilde{g}\}_{Dirac}|_N = \{\tilde{f}, \tilde{g}\}_\Pi|_N$ since $\{\tilde{f}, \varphi^a\}_\Pi|_N = 0$ for all constraints φ^a. $\square$

Now consider an integrable distribution $R \subset TM$ and let D be a Dirac structure on $E_0 \to M$ so that $D \cap R^\circ$ has constant rank[3]. We consider the stretched Dirac subbundle D^{R°.

[3]Here $R^\circ \subset T^*M$ denotes the annihilator of R, i.e., the sections of R° are the 1-forms that kill all sections of R.

The following proposition shows that in the special case that D is the graph of a Poisson structure Π and the leaves of R are cosymplectic in (M, Π), the Dirac subbundle D^{R° gives exactly the classical Dirac bracket (Def. 5.1). Hence D^{R° can be considered as a generalization of the classical Dirac bracket.

Proposition 5.1. *1) D^{R° is a Dirac structure. It is constructed putting together the integral submanifolds N of R, with the (smooth) Dirac structure induced pulling back D.*

Now assume that D is the graph of a Poisson structure Π.

2a) D^{R° is itself the graph of a Poisson structure iff the leaves of the distribution R are Poisson-Dirac submanifolds [8] *of* (M, Π).

2b) Suppose the stronger condition that the leaves of R are cosymplectic *submanifolds of* (M, Π)*, so that the Dirac bracket* (5.1) *can be defined (see Lemma 5.1). Then D^{R° is the graph of the Poisson structure Π_{Dirac}.*

Proof. 1) Notice that since R is integrable we can choose a frame for R° consisting of closed 1-forms, which act trivially under the Courant bracket. Hence R° is canonical for D (see Def. 4.1). So from Prop. 4.1 we conclude that D^{R° is a Dirac structure.

Since $\pi(D^{R^\circ})$ is everywhere tangent to the foliation R, the integral submanifolds N of R are unions of presymplectic leaves of D^{R°. The Dirac structure D can be restricted to any leaf N of the foliation induced by R, delivering a smooth subbundle since $D \cap R^\circ$ has constant rank [7] [5]. Further, a simple computation shows that the pullback to N of D is equal to the pullback to N of D^{R°.

2a) The Dirac structure D^{R° is the graph of a bivector field if and only if $D^{R^\circ} + TM = TM \oplus T^*M$. Taking orthogonals we obtain $(D + R^\circ) \cap R = \{0\}$. This can be rewritten as $\Pi^\sharp(R^\circ) \cap R = \{0\}$, which by definition [8, Sect. 8] means that the leaves of R are Poisson-Dirac submanifolds of (M, Π).

2b) This follows from Lemma 5.1 ii) and part i) of this Proposition. □

Remark 5.2. Prop. 5.1 2a) shows that even within the framework of Poisson geometry, i.e. in the case that both D and D^{R° correspond to Poisson structures, our construction of Dirac structure D^{R° is more general than the classical Dirac bracket.

5.2. **The Marsden-Ratiu reduction.** We show that the reduced Poisson structure induced via Marsden-Ratiu reduction [13] from a Poisson manifold (M, Π) is obtained pushing forward not Π itself but rather a suitable stretching of Π.

We start by recalling the Poisson reduction by distributions as it was stated by Marsden and Ratiu in [13], see also [15]. The set-up is the following:

$(M, \{\cdot,\cdot\})$ is a Poisson manifold,
N is a submanifold with embedding $\iota : N \hookrightarrow M$,
$B \subset T_N M$ is a smooth subbundle of TM restricted to N.

We shall also assume that $F := B \cap TN$ is an integrable regular distribution on N and $\underline{N} := N/F$ is a smooth manifold.

Definition 5.2. [13] $(M, \{\cdot,\cdot\}, N, B)$ is **Poisson reducible** if there is a Poisson bracket $\{\cdot,\cdot\}_{\underline{N}}$ on $\underline{N}$ such that for any $f_1, f_2 \in C^\infty(\underline{N}) \cong C^\infty(N)_F$ we have:

$$\{f_1, f_2\}_{\underline{N}} = \iota^*\{f_1^B, f_2^B\}$$

for all extensions $f_i^B \in C^\infty(M)_B$ of f_i.

Here $C^\infty(N)_F := \{f \in C^\infty(N) \mid \mathrm{d}f|_F = 0\}$ and $C^\infty(M)_B := \{f \in C^\infty(M) \mid \mathrm{d}f|_B = 0\}$.

Given $(M, \{\cdot,\cdot\}, N, B)$ clearly there is at most[4] one Poisson bracket $\{\cdot,\cdot\}_{\underline{N}}$ on $\underline{N}$ satisfying the requirement of Def. 5.2. The following proposition (which is essentially [10, Prop. A.2]) describes the reduced Poisson structure on $\underline{N}$ in terms of bivector fields rather than in terms of brackets: it is obtained from Π by *stretching* along B, pulling back to N and then pushing forward to $\underline{N}$.

Proposition 5.2. *Assume that the prescription of Def. 5.2 gives a well-defined bivector field on $\underline{N}$, denote by $L_{\underline{N}}$ its graph, and denote $L_\Pi = graph(\Pi)$. Then the pullback of the almost Dirac structure $L_{\underline{N}}$ under $p : N \to \underline{N}$ is $\iota^*(L_\Pi^B)$.*

Consequently $L_{\underline{N}}$ is given by the push-forward under p of $\iota^*(L_\Pi^B)$.

5.3. **Couplings on Poisson fibrations.** Given a Dirac subbundle D and an isotropic subbundle S, there are situations in which one wants to "deform" D to a new Dirac subbundle which contains S. A natural candidate for the new Dirac structure is the stretching D^S. An instance is provided by our next example, inspired by the results of Brahic and Fernandes [3] which (in the case of a flat connection) can be rephrased in our formalism.

Our data are a manifold M and:

A splitting of the tangent bundle into two regular, integrable distributions: $TM = Hor \oplus Vert$.
A two form in Hor: $\omega \in \Omega^2(Hor)$.
A bivector field in $Vert$: $\pi_V \in \wedge^2(Vert)$.

The question is how two combine these data and which are the conditions that produce a Dirac structure. In principle there are two dual ways of doing this by using the deformation by stretching. The two different procedures give the same result.

a) Consider $\pi_V^\sharp : T^*M \longrightarrow Vert$ and take $D = \mathrm{graph}(\pi_V^\sharp)$.

D is a Dirac structure if and only if

i) $[\pi_V, \pi_V] = 0$.

To define S, the stretching direction, consider the bundle map $\hat{\omega}^\flat : Hor \longrightarrow Vert^\circ$ induced by ω and take $S = \mathrm{graph}(\hat{\omega}^\flat)$.

S is involutive if and only if:

ii) ω is horizontally closed,

iii) $\mathcal{L}_v(\omega(u_1, u_2)) = 0$ for $v \in \Gamma(Vert)$ and $u_i \in \Gamma(Hor)$ s. t. $[v, u_i] \in \Gamma(Vert)$.

[4]References [13], and subsequently [10], formulate conditions which ensure that $(M, \{\cdot,\cdot\}, N, B)$ is Poisson reducible.

Now, assuming that the other conditions hold, we can show that S preserves D^S if and only if
$iv)$ $\mathcal{L}_u \pi_V = 0$ for any $u \in \Gamma(Hor)$ s. t. $[v,u] \in \Gamma(Vert)$, $\forall\ v \in \Gamma(Vert)$.

If conditions $i)$-$iv)$ are satisfied then, using Thm. 4.1 ($S^\perp = S + Vert + Hor^\circ$ and therefore $\pi(S^\perp) = TM$), we have that D^S defines a Dirac structure. In the next paragraph we shall show an alternative way of obtaining the same result with the roles of ω and π_V exchanged.

b) We introduce first the bundle map $\omega^\flat : Hor \longrightarrow Hor^*$.

Consider the Dirac subbundle $D' \subset TM \oplus T^*M$ induced by graph($\omega^\flat$), i.e.

$$D' = \{(v,\xi)|v \in Hor, \xi|_{Hor} = \omega^\flat v\}.$$

One can show that D' is a Dirac structure if and only if condition $ii)$ above holds.

Now we proceed to define the new stretching subbundle S'. Take

$$S'_x = \{((\pi_V \xi)_x, \xi_x)|\xi_x \in (Hor^\circ)_x\}.$$

S' is involutive if and only if conditions $i)$ and $iv)$ hold.

Finally one can show that, assuming all previous conditions, the stretching $D'^{S'}$ is a Dirac structure if and only if
$iii)'$ $\mathcal{L}_v(\omega(u_1,u_2)) = 0$ for $v \in \Gamma(\pi_V^\sharp(T^*M))$ and $u_i \in \Gamma(Hor)$ s. t. $[v,u_i] \in \Gamma(Vert)$.

It is interesting to compare the two construction. First it is clear that $D^S = D'^{S'}$. Further, condition $iii)$ in construction a) implies condition $iii)'$ in b). Therefore, even if both give the same final result, the second construction has a broader range of application.

Remark 5.3. We establish the connection between the above and the coupling of Poisson fibrations of ref. [3]. Suppose that M is the total space of a fibration so that $Vert$ is the distribution tangent to the fibers. One computes easily that D^S agrees with the fiber non-degenerate almost Dirac structure associated to the triple (π_V, Hor, ω) in Cor. 2.6 of [3]. Brahic and Fernandes compute the necessary and sufficient conditions for this to be a Dirac structure in Cor. 2.8 of [3]. If the horizontal connection is flat their conditions are equivalent to our $i), ii), iii)'$ and $iv)$ above, i.e. the conditions for having a Dirac structure following the stretching procedure introduced in the paper.

Acknowledgments: Research partially supported by grants FPA2003-02948 and FPA2006-02315, MEC (Spain) and by the Forschungskredit 2006 of the University of Zürich.

References

[1] G. Blankenstein and T. S. Ratiu, *Singular reduction of implicit Hamiltonian systems*, Rep. Math. Phys. **53** (2004), 211–260.

[2] G. Blankenstein and J. van der Schaft, *Symmetry and reduction in implicit generalized Hamiltonian systems*, Rep. Math. Phys. **47** (2001), 57–100.

[3] O. Brahic and R.L. Fernandes, *Poisson Fibrations and Fibered Symplectic Groupoids*, Contemporary Mathematics **450** (2008), 41–60.
[4] H. Bursztyn, G. Cavalcanti and M. Gualtieri, *Reduction of Courant algebroids and generalized complex structures*, Adv. Math. **211**, no. 2 (2007), 726–765.
[5] H. Bursztyn, A. Weinstein *Poisson geometry and Morita equivalence* London Math. Soc. Lecture Note Ser. **323** (2005), 1–78.
[6] I. Calvo and F. Falceto, *Reduction and projection of Dirac structures*, Monografías de la Real Academia de Ciencias de Zaragoza **29** (2006), 49-56.
[7] T. Courant, *Dirac manifolds*, Trans. Amer. Math. Soc. **319** (1990), 631–661.
[8] M. Crainic and R. Fernandes, *Integrability of Poisson brackets*, J. Differential Geom. **66**, no. 1 (2004), 71–137.
[9] P. A. M. Dirac, *Lectures on quantum mechanics*, Yeshiva University, New York, 1967.
[10] F. Falceto and M. Zambon, *An extension of the Marsden-Ratiu reduction for Poisson manifolds*, Lett. Math. Phys. **85**, no. 2-3 (2008), 203–219.
[11] Z. J. Liu, *Some remarks on Dirac structures and Poisson reductions*, Banach Center Publ. **51** (2000), 165–173.
[12] Z. J. Liu, A. Weinstein and P. Xu, *Manin triples for Lie bialgebroids*, J. Differential Geom. **45**, no. 3 (1997), 547–574.
[13] J. E. Marsden and T. Ratiu, *Reduction of Poisson manifolds*, Lett. Math. Phys. **11** (1986), 161–169.
[14] J. E. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys. **5** (1974), 121–130.
[15] J. P. Ortega and T. Ratiu, *Momentum maps and hamiltonian reductions*, Birkhäuser, Basel-Boston-Berlin, 2004.
[16] I. Vaisman, *Isotropic subbundles of* $TM \oplus T^*M$ Int. J. Geom. Methods Mod. Phys. **4**, no. 3 (2007), 487–516.
[17] I. Vaisman, *Weak-Hamiltonian dynamical systems*, J. Math. Phys. **48**, no. 8 (2007).
[18] M. Zambon, *Reduction of branes in generalized complex geometry*, J. Symplectic Geom. **6**, no. 4 (2008), 353–378.

Laboratorio Nacional de Fusión, Asociación EURATOM-CIEMAT, E-28040 Madrid (Spain)
E-mail address: ivan.calvo@ciemat.es

Departamento de Física Teórica and Instituto de Biocomputación y Física de Sistemas Complejos, Universidad de Zaragoza, E-50009 Zaragoza (Spain)
E-mail address: falceto@unizar.es

Universidade do Porto, Departamentos de Matematica Pura, Rua do Campo Alegre 687, 4169-007 Porto (Portugal)
E-mail address: mzambon@fc.up.pt